Abstract
Some further theoretic properties of the scheme called $\Gamma_1$ non-deranged permutation Group, especially in relation to Descent were identified and studied in this paper. This was done by first computations on this scheme using prime numbers $p \geq 5$. A recursion formula for generating the Descent number, union of Descent set, Descent bottom sum and Descent top sum was developed and used these numbers to identify theoretic consequences.

Keywords: Descent Numbers; Descent set; $\Gamma_1$ non-deranged permutation Group.

1 Introduction
A Permutation $f$ of the $\Gamma_1$-non deranged permutation group presents as descent in $i$ whenever $f(i) > f(i + 1)$. Permutation statistics were first introduced by Euler(1913) and then extensively studied by MacMahon (1915). In the last decades much progress has made, both in the discovery and the study of new statistics, and in extending these to other type of permutations such as words and restricted permutation. The concept of derangements in permutation groups (that is permutations without a fix element) has proportion in the underlying symmetric group $S_n$. Garba and Ibrahim (2010) used the concept to develop a scheme for prime numbers $p \leq 5$ and $\Omega \subseteq N$ which generate the cycles of permutations (derangements) using $\omega_i = ((1)(1+i)_{mp}(1+2i)_{mp}...(1+(p-1)i)_{mp})$ to determine the arrangements. It is difficult for a set of derangements to be a permutation group because of the absence of the natural identity element (a non derangement). The construction of the generated set of permutations from the work of Garba and Ibrahim (2010) as a permutation group was done by Usman and Ibrahim (2011). hey achieved this by embedding an identity element into the generated set of permutation(strictly derangements) with the natural permutation composition as the binary operation (the group was denoted as $G_p$) there is no doubt, the patterns in permutations have been well studied for over a century. It seems to be the case, these patterns were studied on permutations arbitrary. he symmetric group $S_n$ is the set of all permutations of a set $\Gamma$ of cardinality $n$. There are several types of other smaller permutation groups (subgroup of $S_n$) of set $\Gamma$, a notable one among them is the alternating group $A_n$. On the other hand, $\Gamma_1$-non deranged permutations has been established as a group by Aremu et al. (2016), in their work, they studied the abstract theoretical properties of this $\Gamma_1$-non deranged permutation group and further established that the permutation group $G_p^{\Gamma_1}$ is a subgroup of $S_p$ ($p$ is a prime number). Afterwards, Ibrahim et al. (2016) studied the representation of $\Gamma_1$-non deranged permutation group $G_p^{\Gamma_1}$ via group character, hence
established that the character of every \( \omega \in G_p^{\Gamma_1} \) is never zero. Also the non standard Young tableaux of \( \Gamma_1 \)-non deranged permutation group \( G_p^{\Gamma_1} \) has been studied by Garba et al.(2017), they established that the Young tableaux of this permutation group is non standard. Aremu et al.(2017a) studied pattern popularity in \( \Gamma_1 \)-non deranged permutations they establish algebraically that pattern \( \tau_1 \) is the most popular and pattern \( \tau_3, \tau_4 \) and \( \tau_5 \) are equipopular in \( G_p^{\Gamma_1} \) they further provided efficient algorithms and some results on popularity of patterns of length-3 in \( G_p^{\Gamma_1} \). Aremu et al.(2017b) studied Fuzzy on \( \Gamma_1 \)-non deranged permutation group \( G_p^{\Gamma_1} \) and discover that, it is a one sided fuzzy ideal (only right fuzzy but not left) also the \( \alpha \)-level cut of \( f \) coincides with \( G_p^{\Gamma_1} \) if \( \alpha = \frac{1}{p} \).

Ibrahim et al.(2017) studied Ascent on \( \Gamma_1 \)-non deranged permutation group \( G_p^{\Gamma_1} \) and discover that, the union of Ascent of all \( \Gamma_1 \)-non derangement is equal to identity also observed at the difference between \( \text{Asc}(\omega_i) \) and \( \text{Dsc}(\omega_{p-1}) \) is one. Hence we will in this paper developed a recursion formula for generating the Descent number, union of descent set and intersection of descent set and used these numbers to identify theoretic consequences.

2.0 PRELIMINARIES

Definition 2.1
Let \( \Gamma \) be a non empty set of prime cardinality greater or equal to 5 such that \( \Gamma \subset \mathbb{Z} \) bijection \( \omega \) on \( \Gamma \) of the form
\[
\omega_i = \begin{bmatrix} 1 & 2 & 3 & \ldots & p \\ 1 + i \pmod{p} & (1 + 2i) \pmod{p} & (1 + (p-1)i) \pmod{p} \end{bmatrix}
\]
is called a \( \Gamma_1 \)-non deranged permutation. We denoted \( G_p \) to be the set of all \( \Gamma_1 \)-non deranged permutations.

Definition 2.2
The pair \( G_p \) and the natural permutation composition forms a group which is denoted as \( G_p^{\Gamma_1} \). This is a special permutation group which fixes the first element of \( \Gamma \).

Definition 2.3
A descent of permutation \( f = \{1, 2, 3, \ldots, n\} \) is any positive \( i > n \) (where \( i \) and \( n \) are positive integers) where the current value is less than the next, that is \( i \) is an descent of a permutation \( f(i) > f(i+1) \). The descent set of \( f \), denoted as \( \text{Des}(f) \), is given by \( \text{Des}(f) = \{i : f(i) > f(i+1)\} \) the descent number of \( f \), denoted as \( \text{des}(f) \), is defined as the number of descent and is given by \( \text{des}(f) = |\text{Des}(f)| \).

Lemma 2.4
It follows from the properties of integer modulo that for \( n \in \mathbb{Z} \) an integer \( n \pmod{n} = 0 \) also implies \( n = 0 \).

**Proof**
From definition of integer modulo, we know that
\[
\boxed{n \equiv 0} \quad \text{(1)}
\]
It also follows from property of additive identity of integers hat
\[
0 \equiv 0 \quad \text{(2)}
\]
Now, equation (1) implies \( \frac{n}{n} = 1 + 0 \)
equation (2) implies \( \frac{0}{n} = 0 + 0 \)

That is
\[
\frac{n}{n} = 1
\]
Is modulo equivalent to
\[
\frac{0}{n} = 0
\]
That is
\[
\frac{n}{n} = 0 \quad \text{or} \quad \frac{0}{n} = 0
\]
where \( \equiv \) represents a notation modulo equivalence.
It follows \( \equiv \) is an equivalence relation as we shall establish in the following claim.

**Claim**
If \( \frac{n}{n} = 0 \)
Then \( \equiv \) is an equivalence relation.

**Reflexive**
\[
\frac{n}{n} \equiv \frac{n}{n}
\]
By definition i.e uses \( 1 + 0 \equiv 1 + 0 \)
If \( \frac{n}{n} = 0 \)
Then using modulo \( \frac{n}{n} = 1 + 0 \)
And \( \frac{0}{n} = 0 + 0 \)
In any case both \( \frac{n}{n} \quad \text{and} \quad \frac{0}{n} \equiv 0 \)
i.e
\[
\frac{n}{n} \equiv \frac{0}{n}
\]

**Transitive**
If \( \frac{n}{n} \equiv \frac{0}{n} \)
And \( \frac{0}{n} \equiv \frac{m}{n} \)
It implies that \( \frac{n}{m} = \frac{m}{n} \)

\[ \frac{n}{n} \equiv 0 \quad \frac{n}{n} \equiv 0 \]

It implies that \( \frac{0}{n} = 0 \)

And \( \frac{0}{m} = \frac{m}{n} \)

It implies that \( \frac{m}{n} = 0 \)

It follows \( \frac{n}{n} = 0 \)

From the claim it follows \( n = 0 \) whenever \( n \) is the integer modulo.

For the remainder in this paper we shall adopt the notation established in lemma 2.4 that \( n = 0 \) for all \( n \) representing the modulus of sub-sequences in given permutation

\[ \omega_i = \left( \frac{1}{(1+i)_{mp}} \frac{2}{(1+2i)_{mp}} \ldots \frac{p}{(1+(p-1)i)_{mp}} \right) \]

In particular we shall let \( n = p \) for some prime integer \( p \geq 5 \).

### 3.0 MAIN RESULTS

In this section, we present some descent results of subgroup \( G_p^\Gamma_i \) of \( S_p \) (Symmetry group of prime order with \( p \geq 5 \)).

**Theorem 3.1**

Let \( G_p^\Gamma_i \) be a \( \Gamma_1 \)-non derangement permutations, The total number of descent is

\[ des(G_p^\Gamma_i) = \frac{1}{2} (p-1)(p-2) \]

**Proof**

The order of \( G_p^\Gamma_i \) and the number of positions with ascent or descent are both \( p-1 \). We know that total number of ascent is greater than the total number of descent by \( p-1 \) in \( G_p^\Gamma_i \) from the proposition that says \( asc(G_p^\Gamma_i) - des(G_p^\Gamma_i) = p-1 \) and from the theorem that say \( asc(G_p^\Gamma_i) = \frac{1}{2} p(p-1) \). Then

\[ des(G_p^\Gamma_i) = \frac{1}{2} p(p-1) - (p-1) \]

\[ = \frac{1}{2} (p-1)(p-2) \]

**Proposition 3.2**

If the ascent set of a \( \Gamma_1 \)-non derangement \( \omega_i \) is \( X \subseteq \Gamma \), then the descent set of \( \omega_{p-1} = X - \{1\} \)

**Proof**
If \( \theta_i, \theta_j \in S_p \) (where \( i \neq j \) and \( i, j \in \mathbb{Z} \)) then the ascent of \( \theta_j \) is equal to the descent set of \( \theta_j \). By the non derangement property of \( G_p^{\Gamma_1} \) is not contained in the descent set of any arbitrary \( \omega \in G_p^{\Gamma_1} \) and so there exist a set \( X \setminus \{1\} \) that will be descent set of an arbitrary \( \Gamma_1 \)-non derangement \( \omega_{p-1} \).

**Lemma 3.3**
Suppose that \( G_p^{\Gamma_1} \) is \( \Gamma_1 \)-non deranged permutations. Then
\[
\text{Des}(\omega) \cap \text{Des}(\omega_{p-1}) = \emptyset
\]

**Proof.**
Suppose \( \omega = a_1a_2\ldots a_{p-1}a_p \) and \( \omega_{p-1} = a_1a_p a_{p-1}\ldots a_2 \). By restricting \( a_1 \) since it is the least of all functions and it is at the first position of \( \omega \) and \( \omega_{p-1} \), that is, it has no effect on the descent, so we have that
\[
\text{Des}(\omega) = \text{Asc}(\omega_{p-1}),
\]
(1)
And
\[
\text{Asc}(\omega) = \text{Des}(\omega_{p-1}).
\]
(2)
It is obvious from the definition of descent ascent that
\[
\text{Des}(\omega) \cap \text{Asc}(\omega) = \emptyset
\]
(3)
Substituting (2) into (3) we have
\[
\text{Des}(\omega) \cap \text{Des}(\omega_{p-1}) = \emptyset
\]

**Lemma 3.4**
Suppose that \( G_p^{\Gamma_1} \) is \( \Gamma_1 \)-non deranged permutations. Then
\[
\text{Des}(\omega) \cup \text{Des}(\omega_{p-1}) = \text{Des}(\omega_{p-1})
\]

**Proof.**
Given \( \omega = a_1a_2\ldots a_{p-1}a_p \), then \( \omega_{p-1} = a_1a_p a_{p-1}\ldots a_2 \). By restricting \( a_1 \) because it has no effect on the descent since it is the least of all functions and it is at the first position in \( \omega \) and \( \omega_{p-1} \). Since \( \omega_{p-1} \) is a strictly decreasing sequence when \( a_1 \) is restricted.
\[
\text{Des}(\omega_{p-1}) = \{i : f(i) < f(i + 1)\} \cup \{i : f(i) > f(i + 1)\}
\]
\[
= \text{Des}(\omega) \cup \text{Asc}(\omega)
\]
\[
= \text{Des}(\omega) \cup \text{Des}(\omega_{p-1})
\]
By this we can see that
\[
\text{Des}(\omega) \cup \text{Des}(\omega_{p-1}) = \text{Des}(\omega_{p-1})
\]

**Corollary 3.5**
Suppose that \( G_p^{\Gamma_1} \) is \( \Gamma_1 \)-non deranged permutations. Then \( \text{des}(\omega) + \text{des}(\omega_{p-1}) = \text{des}(\omega_{p-1}) \)

**Proof**
Since by lemma 3.4, \( \text{Des}(\omega_i) \cup \text{Des}(\omega_{p^{-1}i}) = \text{Des}(\omega_{p^{-1}i}) \) and by lemma 3.3, \( \text{Des}(\omega_i) \cap \text{Des}(\omega_{p^{-1}i}) = \emptyset \), the result follows.

**Theorem 3.6**

Let \( G_p^{\Gamma_1} \) be a \( \Gamma_1 \)-non derangement permutations then \( \bigcup_{i=1}^{p^{-1}} \text{Des}(\omega_i) = \text{Des}(\omega_{p^{-1}i}) \).

**Proof**

Since \( \omega_{p^{-1}i} = 1p(p-1)...2 \) so it takes all possible descent in \( \omega_i \), but by lemma 3.4, \( \text{Des}(\omega_i) \cup \text{Des}(\omega_{p^{-1}i}) = \text{Des}(\omega_{p^{-1}i}) \), we want to show that for any \( G_p^{\Gamma_1} \) there exist \( \omega_i \) and \( \omega_{p^{-1}i} \) where \( i \neq 1 \). Since \( P \geq 5 \) and any \( G_p^{\Gamma_1} \) consist of non-deranged permutations \( \{\omega_1, \omega_2, ..., \omega_{p^{-1}i}\} \), from this at least \( \omega_1 \) and \( \omega_{p^{-1}i} \).

**Theorem 3.7**

Let \( G_p^{\Gamma_1} \) be a \( \Gamma_1 \)-non derangement permutations then \( \bigcap_{i=1}^{p^{-1}} \text{Des}(\omega_i) = \emptyset \).

**Proof**

From lemma 3.3, \( \text{Des}(\omega_i) \cap \text{Des}(\omega_{p^{-1}i}) = \emptyset \), we want to show that for any \( G_p^{\Gamma_1} \) there exist \( \omega_i \) and \( \omega_{p^{-1}i} \) since \( P \geq 5 \) and any \( G_p^{\Gamma_1} \) consist of non-deranged permutations \( \{\omega_1, \omega_2, ..., \omega_{p^{-1}i}\} \), does not have any element in common the result follows.

**4.0 CONCLUSION**

This paper has provided very useful theoretical properties of this scheme called \( \Gamma_1 \)-non deranged permutations in relation to the Descent. We have shown that the intersection of Descent set of all \( \Gamma_1 \)-non derangement is empty, we also observed that the descent number is strictly less than Ascent number by \( P - 1 \).

**5.0 RECOMMENDATION**

Further researches should be conducted on \( \Gamma_1 \)-non deranged permutations in relation to the other permutation statistics such as Excedance, Inversion, Major index, Rise. In order find new algebraic and combinatorial results.

**References**

Aremu K.O., Garba A.I., Ahmad R., Ejima O. and Usman H. (2016), Abstract group theoretical properties of \( \Gamma_1 \)-non deranged permutation group \( G_p^{\Gamma_1} \). *Nigerian Journal of Basic and Applied Sciences*, (accepted).
