ON THE NUMERICAL SOLUTION OF DIFFERENTIAL-ALGEBRAIC EQUATIONS (DAEs) OF HIGHR INDEX-3 USING RALSTON METHOD

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Abstract
In this paper, Numerical solution of differential-algebraic equations of higher index specifically index-3 is considered by Ralston's method. We applied this method to two numerical examples and solution have been compared with those obtained by variational Iteration method and exact solutions. The numerical error estimated shows that the Ralston's Method confirms better than variational iteration method of the same order. The Ralston's Method have better stability properties.

Keywords: *Differential -Algebraic Equations (DAEs), Index-3, Ralston's Method, Heisenberg form of differential algebraic- equations.*

Introduction
Differential-Algebraic Equations (DAEs) can be found in a wide variety of Scientific and engineering applications. It has attracted considerable attention in recent years, considerable efforts have been made to solve systems of DAEs. Many important mathematical models and physical problems are mostly initially modeled as a system of Differential-Algebraic Equations(DAEs). Some numerical methods have been developed using backward differentiation formulae (BDF), Pade approximations method. These methods are only suitable for low index problems. However, researcher kept improving on the solution to DAE problem of both lower and higher index using different methods whose important cannot be over emphasize. The most general form of DAEs is given by

$F(t, x, x') = 0 \ldots (1)$

Where $\partial F/\partial x'$ may be singular. The rank and structure of the jacobian matrix may depend, in general on the solution $x(t)$ and for simplicity we will always assume that it is independent of t. The important special case of a semi-explicit DAE or an ODE with constrains,

$x' = f(t, x, z) \ldots (2a)$

$0 = g(t, x, z) \ldots (2b)$

This is a special case of (1). The index is 1 if $\partial g/\partial z$ is non-singular, because the first derivative of (2) yield z' in principle. For the semi-explicit index-1 DAE we can distinguish between differential variables $x(t)$ and algebraic variables$z(t)$.The algebraic variables may be less smooth than the differential variables by one derivative. In general case, each component of x may contain a mix of differential and algebraic components, which makes the numerical solution of such high-index problems much more difficult.Most higher- index problems encountered in practice can be expressed as a combination of restrictive structure of ODEs coupled with constraints. In such systems the algebraic and differential variables are identified explicitly for higher-index DAEs and the algebraic variables may all be eliminated using same number of differentiation. And these is called Heisenberg form. There are many excellent and exhaustive texts on this subject that may be consulted such as Acher,U.M (1995),Brenan,K.E (1989),Boyce,W (1986),Fatokun,J.O (2011),Petzold,L.R (1986) and Yelogu,T (2008),There are many new publication in the field of analytical survey such as (Ascher,U.M et al 1991; Cash,J.R 1976; Ganji D.D.et al 2011;).The Variational Iteration Method (VIM) was developed by (Wazwar, A.M.2009). The method has been proved by many author's (Karimi V.S et al 2009;Melik,k and Ercan,C 2022).

The non-linear semi-explicit index-3 systems is a system of the form

$$F(x', x, y, z, t) = 0 \ ...(3a)$$
$$f_2(x, y, t) = 0 \ \ldots \ (3b)$$

$f_3(x, t) = 0 \ \ldots$ (3c) is called Hessenberg Index-3

Where $\partial F/\partial x'$ is non singular, by nothing that the first equation in (3) can be solve for x' to obtain the systems of the form

$$x' + g_1(x, y, z, t) = 0 \ ... (4a)$$
$$g_2(x, y, t) = 0 \ ...(4b)$$
$$g_3(x, t) = 0 \ ...(4c)$$

Where $\partial g_2/\partial y$ and $[\left(\frac{\partial g_3}{\partial x}\right)\left(\frac{\partial g_1}{\partial z}\right)]$ are both non singular.

The term algebraic in the context of DAEs only means free derivative and is not related to abstract algebra. Numerical solution of higher index systems can be sensitive small perturbation, such as those introduced by round off error. We shall extend the result for order, and stability properties of Ralston's method from index -1 system to index -3 differential algebraic system. The choice of using Ralston's method is due to its advantage of having higher order and good stability properties for the numerical solution of differential algebraic system.

The Method

Ralston's Method is a second order Runge-kutta method that allows us to take advantage of not calculating $f'(x_i y_i)$

Given a non-linear index-3 system of the form

$x' + g_1(x, y, z, t) = 0$, $g_2(x, y, t) = 0$, $g_3(x, t) = 0 \ \ldots(2)$.

Consider the second order M-stage Runge-Kutta $F\left(y_{n-1} + h\sum_{j=1}^{M} a_{ij}\, Y'_j, Y'_i, t_{n-1} + c_{ih}\right) =$

$$0, y_n = y_{n-1} + h\sum_{i=1}^{M} b_i Y'_i$$
$$i = 1,2, \ldots M, h = t_n - t_{n-1}$$

Apply to (2)

$$X'_i + g_1\left[x_{n-1} + h\sum_{j=1}^{M} a_{ij}X'_{j,}y_{n-1} + h\sum_{j=1}^{M} a_{ij}Y'_j, z_{n-1} + h\sum_{j=1}^{M} a_{ij}\, Z'_j, t_i\right]$$

$Y'_i + g_2\left[x_{n-1} + h\sum_{j=1}^{M} a_{ij}X'_j\, , y_{n-1} + h\sum_{j=1}^{M} a_{ij}\, Y'_j, t_i\right]$=0

$g_3\left[x_{n-1} + h\sum_{j=1}^{M} a_{ij}X'_j, t_i\right] = 0, i = 1,2, \dots M, x_n = x_{n-1} + h\sum_{i=1}^{M} b_i\, X'_{i,} y_n = y_{n-1} + h\sum_{i=1}^{M} b_i\, Y'_i$

$z_n = z_{n-1} + h\sum_{i=1}^{M} b_i Z'_i$ (3)

Where $t_i = t_{n-1} + c_{ih}$.

Supposing $X_i = x_{n-1} + h\sum_{j=1}^{M} a_{ij}X^i{}_j, Y_i = y_{n-1} + h\sum_{j=1}^{M} a_{ij}\, Y^i{}_j, Z_i = z_{n-1} + h\sum_{J=1}^{M} a_{ij}Z^i{}_j$ (4)

and that the method has an order $k_{I,}$, satisfies strict stability condition and the true solution satisfies;

$$x'(t_i) + g_1[x(t_i), y(t_i), z(t_i), t_i] = 0$$
$$y'(t_i) + g_2[x(t_i), y(t_i), t_i] = 0$$
$$g_3[x(t_i), t_i] = 0, i = 1,2,3, \dots M$$
$$x(t_n) = x(t_{n-1}) + h\sum_{i=1}^{M} b_i x'(t_{n-1} + c_i h) - \delta_{M+1}^{x(n)}$$

$y(t_n) = y(t_{n-1}) + h\sum_{i=1}^{M} b_i\, y'(t_{n-1} + c_i h) - \delta_{M+1}^{y(n)} + z(t_n) = z(t_{n-1}) + h\sum_{i=1}^{M} b_i z'(t_{n-1} + c_i h) - \delta_{M+1}^{z(n)}$ ….(5)

$$z(t_i) = z(t_{n-1}) + h\sum_{j=1}^{M} a_{ij} z'(t_{n-1} + c_j h) - \delta_i^{z(n)} i = 1,2,3, \dots, M$$

Subtracting (5) from (3) we have

$e_n^x = e_{n-1}^x + h\sum_{i=1}^{M} b_i E_i^{x'} + \delta_{M+1}^{y(n)}$ ………………. (6)

$e_n^y = e_{n-1}^y + h\sum_{i=1}^{M} E_i^{y'} + \delta_{M+1}^{y(n)} \dots\dots\dots\dots\dots\dots\dots..$ (7)

$e_n^z = e_{n-1}^z + h\sum_{i=1}^{M} b_i\, E_i^{z'} + \delta_{M+1}^{z(n)}$ ……………………..…(8)

Let $G_{31}(t_i) = \frac{\partial_{g3}}{\partial x}, G_{21}(t_i) = \frac{\partial_{g2}}{\partial x}, G_{22}(t_i) = \frac{\partial_{g2}}{\partial x}, G_{11}(t_i) = \frac{\partial_{g1}}{\partial x}, G_{12}(t_i) = \frac{\partial_{g1}}{\partial y}, G_{13}(t_i) = \frac{\partial_{g1}}{\partial z}$.

Where the partial derivatives are evaluated along the true solutions at t_i.following the procedure for the subtraction above, we obtain

$E_i^{x'} + G_{11}(t_i)E_i^x + G_{12}(t_i)E_i^y + G_{13}(t_i)E_i^z = \eta_i^x$ … (9)

$E_i^{y'} + G_{21}(t_i)E_i^x + G_{22}(t_i)E_i^y = \eta_i^y$ … (10)

$G_{31}(t_i)E_i^x = \eta_i^z$ … (11)

where

$$e_n^x = e_{n-1}^x + h\sum_{i=`1}^{M} b_i\, E_i^{x'} + \delta_{M+1}^{x(n)}$$
$$e_n^y = e_{n-1}^y + h\sum_{i=1}^{M} b_i\, E_i^{y'} + \delta_{M+1}^{y(n)}$$
$$e_n^z = e_{n-1}^z + h\sum_{i=1}^{M} b_i E_i^{z'} + \delta_{M+1}^{z(n)}$$

$$E_i^x = e_{n-1}^x + h\sum_{j=1}^{M} a_{ij} E_j^{x\prime} + \delta_n^{x(n)}$$

$E_i^y = e_{n-1}^y + h\sum_{j=1}^{M} a_{ij}\, E_j^{y\prime} + \delta_i^{y(n)}$, (12)

$$E_i^z = e_{n-1}^z + h\sum_{j=1}^{M} a_{ij}\, E_j^{z\prime} + \delta_i^{z(n)}$$

η_i terms are higher order terms inE_i^x, E_i^y, E_i^z . Multiplying the first equation in (11) by $G_{31}(t_i)$ *and solving for* E_i^z and by letting $M_i = G_{31}(t_i)[G_{13}(t_i)G_{31}(t_i)]^{-1}$ we have that

$E_i^z = -M_i E_i^{x\prime} - M_i G_{11}(t_i)E_i^x - M_i G_{12}(t_i)E_i^y + M_i\eta_i^x$ ... (13)

Suppose $H_i = G_{13}(t_i)M_i$, multiplying the first equation in (11) by $I - H_i$ and substituting equation (13) for E_i^z then we have

$(I - H_i)E_i^{x'} + N_i E_i^x + D_i E_i^y + (I - H_i)M_i G_{11}(t_i)E_i^x - G_{13}(t_i)M_i G_{12}(t_i)E_i^y +$
$G_{13}(t_i)M_i\eta_i^x = \eta^{\sim}{}_i^x$ (14)

By applying the definition of M_i in (14), we have

$(I - H_i)E_i^{x'} + N_i E_i^x + D_i E_i^y + [I - H_i]\left[-E_i^{x'} - G_{11}(t_i)E_i^x - G_{12}(t_i)E_i^y + \eta_i^x\right] = \eta^{\sim}{}_i^x$
…(15)

Remark: the above expressions have been reduce by eliminating E_i^z which by implication, also reduce the index to index 2.

Results

Numerical example 1: Consider the following index-3 differential –algebraic equations with hessenbergindex-3 of the form

$x'_2 + x_1 - 1 = 0\,, xx'_2 + x'_3 + 2x_2 = 0\,, xx_2 + x_3 - e^x = 0$ With initial conditions given as

$$x_1(0) = 0, x_2(0) = -1\,, x_3(0) = 1$$

Table 1: Comparison Analysis of Results for Problem 1 for $x_1(x)$

X	**H**	**Exact solution**	**Ralston's Method(RM)**	**Absolute Error (RM)**	**Variational-iteration Method(VIM)**	**Absolute Error(VIM)**
0.0	0	0	0	0	0	0
1.0	0.1	1.718281828	0.105192276	0.9387805	0.1051708	0.9387929
2.0	0.2	6.389056099	0.221577899	0.96531919	0.2214028	0.965346562
3.0	0.3	19.08553692	0.523469832	0.97257602	0.3498508	2.85829591
4.0	0.4	53.59815003	0.493295682	0.99027338	0.4918248	0.990823849
5.0	0.5	147.4131591	0.651663805	0.98784163	0.6487211	0.9995599298
6.0	0.6	402.4287935	0.827324874	0.9979417	0.8221180	0.99795707

7.0	0.7	1095.633158	1.022214129	0.9990701	1.013752707	0.9990747 33
8.0	0.8	2980.957987	1.23846336	0.99958451	1.2255408	0.9995888 76
9.0	0.9	8102.083928	1.478419785	0.99975759	1.4596031	0.9998198 48
10.	1.0	22025.46579	1.744666678	0.9999208	1.7182813	0.9999219 86

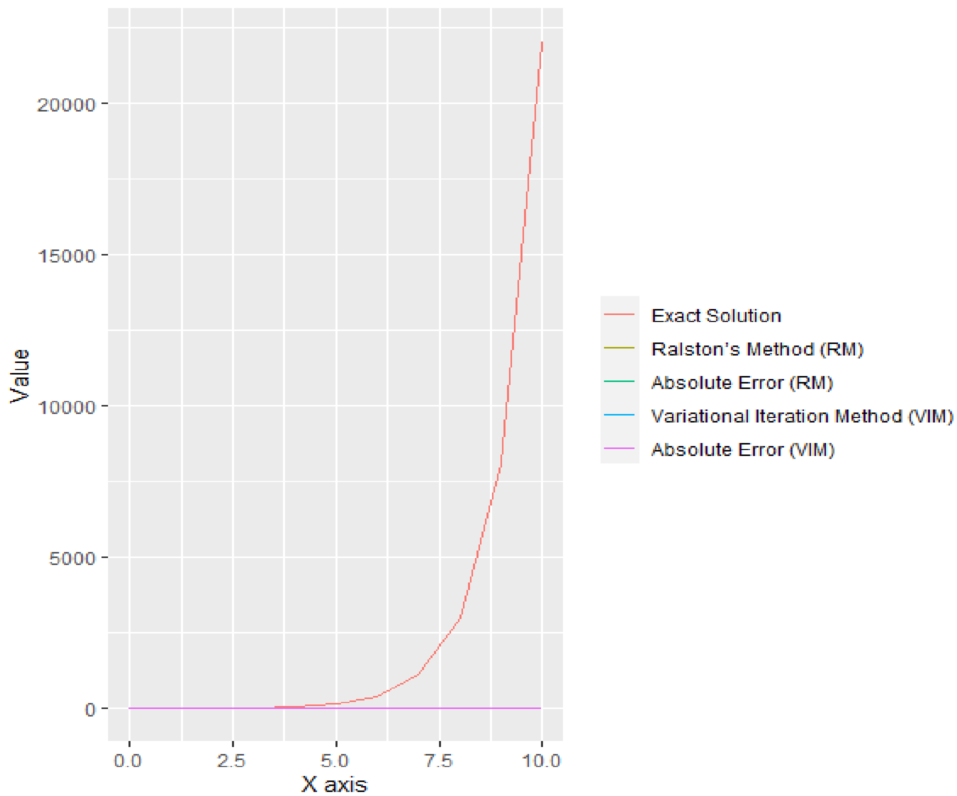

Figure 1: Solution curve showing comparison between Ralston's Method (RM) Variational Iteration Method(VIM) and Exact Solution (ES)

Table 2: Comparison Analysis of Result for Problem 1 for x_2(x)

X	H	Exact Solution	Ralston's Method (RM)	Absolute Error(RM)	Variational iteration Method (VIM)	Absolute Error (VIM)
0.0	0.0	-1	0	1	-1	0
1.0	0.1	0.7182812	0.09480773	1.1319923	-0.90517091	0.260189 082
2.0	0.2	3.3890569	0.17842211	1.0526467	-0.82140275	0.757630 817
3.0	0.3	14.085539	0.24953548	1.0177152	-0.74985880	0.946763 917
4.0	0.4	-46.5981500	0.30670438	1.0065819	-0.69182469	0.985153 387
5.0	0.5	138.41319	0.48336195	1.0034928	-0.64872127	0.995313 153
6.0	0.6	391.42873	0.37267515	1.0095209	-0.62211880	0.998410 646
7.0	0.7	1082.6338	0.37778587	1.0003481	-0.61375270	0.999433 092
8.0	0.8	2964.9578	0.36153669	1.0001217	-0.62554092	0.999789 022
9.0	0.9	8085.0832	0.32158024	1.0000395	-0.65960311	0.999918 417
10.0	1.0	22006.407	0.25533332	1.0000113	-0.71828182	0.999967 36

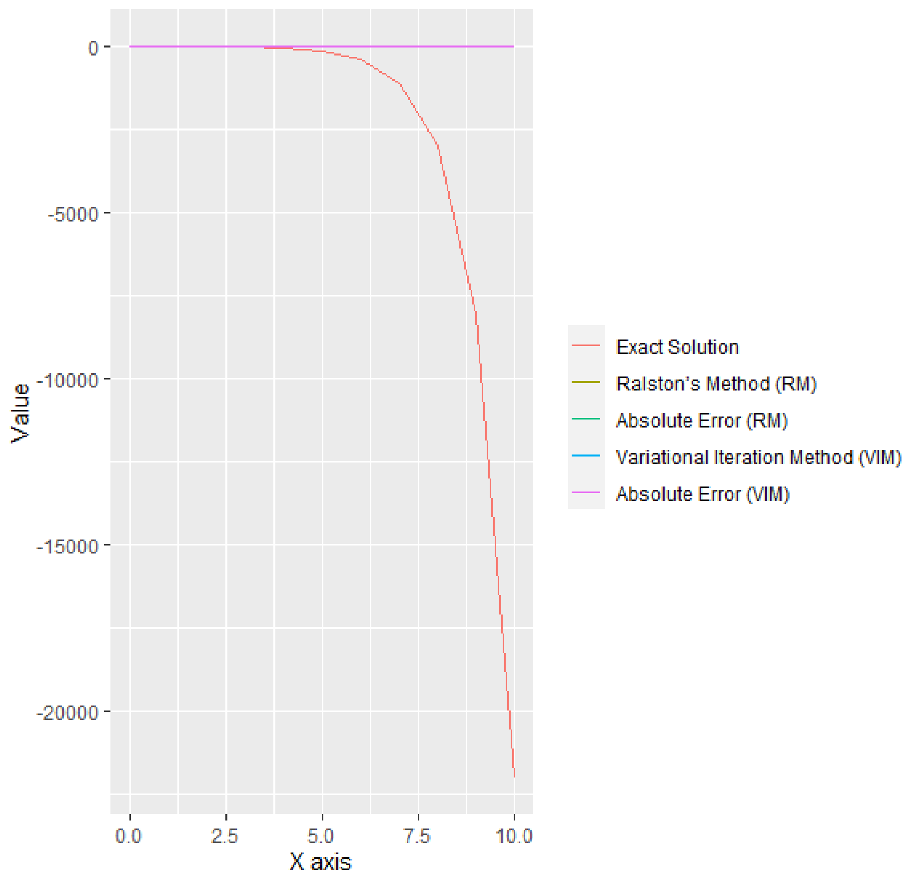

Figure 2: Solution curve showing comparison between Ralston's Method(RM) Variational Iteration Method (VIM) and Exact Solution (ES)

Example2

Consider the problem below

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & x \\ e^x & x+1 & 0 \\ 0 & x^2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x \\ x^2 + x + 2 \\ x^3 \end{pmatrix} \quad (17)$$

With initial conditions as $\begin{pmatrix} x_1(0) \\ x_2(0 \\ x_3(0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Table 3: Comparison Analysis of Result for Problem 2 for the Solution of $x_1\ (x)$

X	H	Exact Solution	Ralston's Method (RM)	Absolute Error(RM)	Variational iteration Method (VIM)	Absolute Error (VIM)
0.0	0.0	1	0	1	1	0
1.0	0.1	0.3678791	-0.09518289	4.8649747	0.90483748	0.536957977
2.0	0.2	0.1353353	-0.18142773	1.7344117	0.81873073	0.68339547
3.0	0.3	0.0497878	-0.25970324	1.1917073	0.74081821	0.691031153
4.0	0	0.0183158	-0.33088485	1.0553509	0.67032006	0.652004408
5.0	0.5	0.0067376	-0.39576309	1.0117021	0.60653069	0.599792713
6.0	0.6	0.0024782	-0.45505126	1.1373007	0.54881166	0.546332884
7.0	0.7	0.0009111	-0.50939250	1.0017904	0.49965850	0.498746649
8.0	0.8	0.0033542	-0.55936620	1.0000598	0.49932894	0.498993502
9.0	0.9	0.0001239	-0.60549385	1.0002035	0.40656969	0.406446281
10.	1.0	0.0000049	-0.64824436	1.0007008	0.36787941	0.367874902

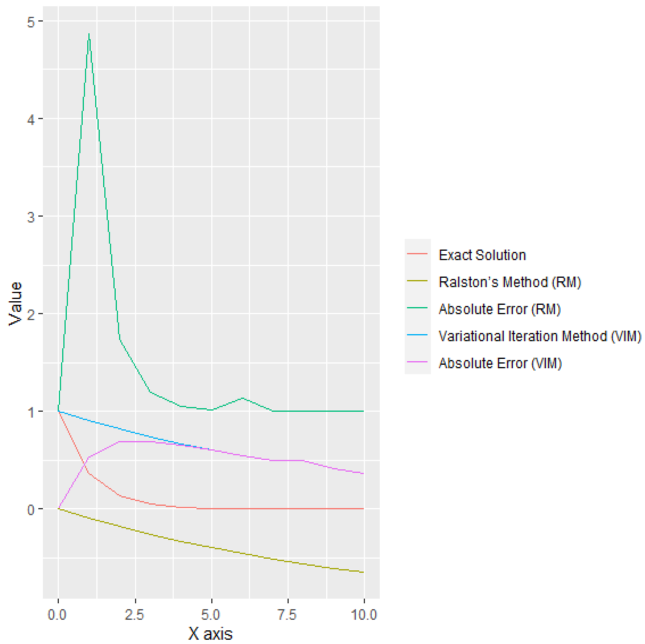

Figure 3: Solution curve showing comparison between Ralston's Method (RM) Variational Iteration Method(VIM) and Exact Solution (ES)

Table 4: Comparison Analysis of Results for Problem 2 for the Solutions of $x_2\,(x)$

X	H	Exact Solution	Ralston's Method	Absolute Error(RM)
0.0	0.0	0	1	1
1.0	0.1	1.0	0.914057745	0.085942255
2.0	02	2.0	0.797688823	0.601155885
3.0	0.3	3.0	0.6473386	0.784220466
4.0	0.4	4.0	0.459083312	0.885229172
5.0	0.5	5.0	0.228594121	0.954281175
6.0	0.6	6.0	0.048902171	1.008150362
7.0	0.7	7.0	0.378666091	1.054095156
8.0	0.8	8.0	0.766492446	1.09581156
9.0	0.9	9.0	1.218761637	1.13541796
10.0	1.0	10.0	0.840381816	0.915961818

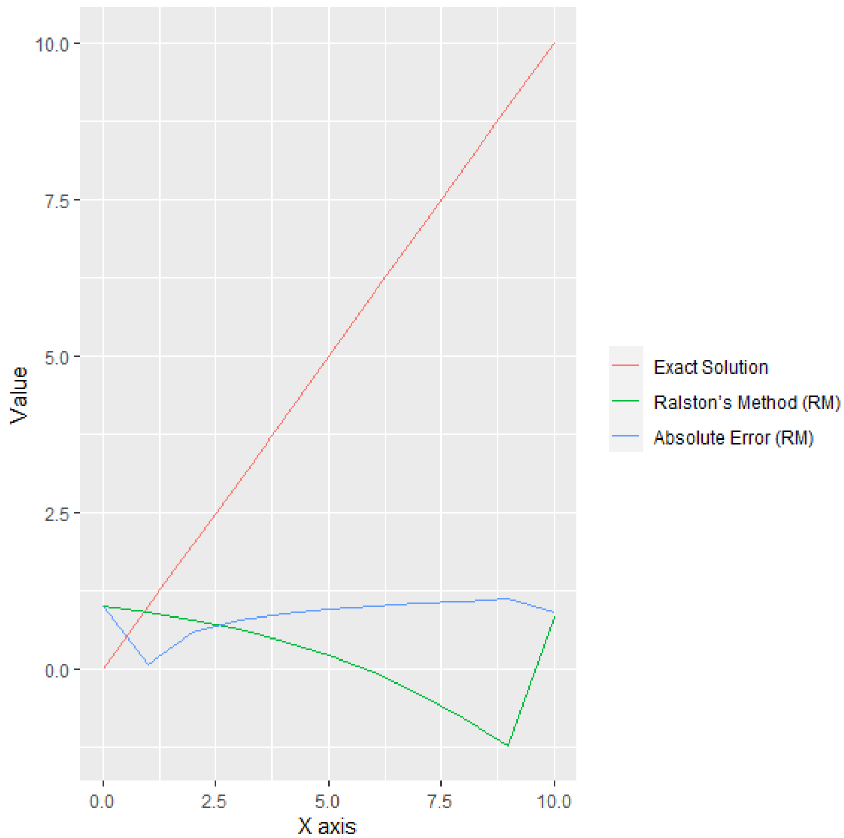

Figure 4: Solution curve showing comparison between Ralston's Method(RM) and Exact Solution (ES)

Discussion

The numerical results presented above, show the implementation of our method when applied to the systems of higher index differential algebraic equations. It is obvious that the method competes favorably with the existing method and exact solution for the class of problems 1 and 2 considered. These can be observed by a close look at table 4 and figure 1 and 2. In Table 2, 3 and 4, the implementation of our method perform better when applied to the class of problem they are designed for as their accuracy are well guarantee. Also, the absolute error estimate obtained from our method compete with the error obtained from variational iteration method. Table 4 above, our method allow the generation of solutions at different step size. In analyzing our results, we apply Ralston method which is a two stage second order Runge- Kutta method on the developed index-3 systems with different step size.

Conclussion

This study has proposed for solving differential-algebraic equations of higher index. Results presented in Table 1-4 and figure 1-4 show that advantages of the method when applied to the systems of higher index differential algebraic equations. It is obvious that the methods compete favorably with the existing method for the class of problems 1 and 2 considered and also yields desire accuracy. This study has contributed to knowledge in the field of numerical analysis through the implementation of our method on systems of differential algebraic equations. The method are easy to implement on higher index problems. Our method enabled cheap absolute error estimate. Therefore, it is recommended for higher index problems.Inspite of the above, the immediate area from the point of view of this study is to advance the method to handle stiff problems since DAEs are general forms of ODEs with constrained.

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